Residual Chiral Symmetry Breaking in Domain-Wall Fermions

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We study the effective quark mass induced by the finite separation of the domain walls in the domain-wall formulation of chiral fermion as the function of the size of the fifth dimension (L_s), the gauge coupling (β) and the physical volume (V). We measure the mass by calculating the small eigenvalues of the hermitian domain-wall Dirac operator ($H_{\rm DWF}(m_0 = 1.8)$) in the topologically-nontrivial quenched $SU(3)$ gauge configurations. We find that the induced quark mass is nearly independent of the physical volume, decays exponentially as a function of L_s, and has a strong dependence on the size of quantum fluctuations controlled by β. The effect of the choice of the lattice gluon action is also studied.

Simulating massless or near-massless fermions on a lattice is a serious challenge in numerical quantum field theory. The origin of the difficulty can be traced to the well-known no-go theorem first shown by Nielsen and Ninomiya, which states that one can not write down a local, hermitian, and chirally-symmetric lattice fermion action without the fermion doubling problem [1]. Hence to have chirally-symmetric fermions on a lattice, one must use nonlocal actions in which the coupling between lattice sites do not identically vanish even when the separation becomes large, as long as one insists on having a 4 dimensional action.

One of the lattice chiral fermion formalisms that have been studied extensively in recent years is the domain-wall fermion, first formulated by D. Kaplan [2] and later modified for realistic lattice simulation by Shamir [3]. In the domain-wall construction, one introduces an extra fifth dimension s with a finite extension. After discretization, the fifth direction has L_s number of lattice sites. If we put the same four-dimensional gauge configuration on every four-dimensional s slices, the five-dimensional massive theory admits a four-dimensional effective theory in which a left-handed chiral fermion lives near the $s = 0$ slice and a right-handed one near $s = L_s - 1$. Integrating out the heavy modes [4], one obtains an effective four-dimensional chiral theory in the limit of $L_s \to \infty$. For finite L_s, however, the two chiral modes can couple to produce an effective quark mass. Strong gauge field fluctuations can induce rather strong coupling, and hence rather large quark mass. This quark mass is expected to decrease exponentially as $L_s \to \infty$ with possible power law corrections.

In Ref. [5], we proposed to measure the induced effective quark mass by considering the eigenvalue of the hermitian domain-wall Dirac operator $H_{\rm DWF}(m_0) = R_s \gamma_5 D_{\rm DWF}(m_0)$, where R_s denotes the reflection in the fifth direction. In the $L_s \to \infty$ limit, the lattice version of the Atiyah-Singer theorem [6] guarantees the existence of exact zero modes in the presence of a instanton background. For finite L_s, the lowest eigenvalues of the Dirac operator are not zero. We take the average of these would-be-zero eigenvalues as the effective quark mass. The result is qualitatively consistent with those obtained from other

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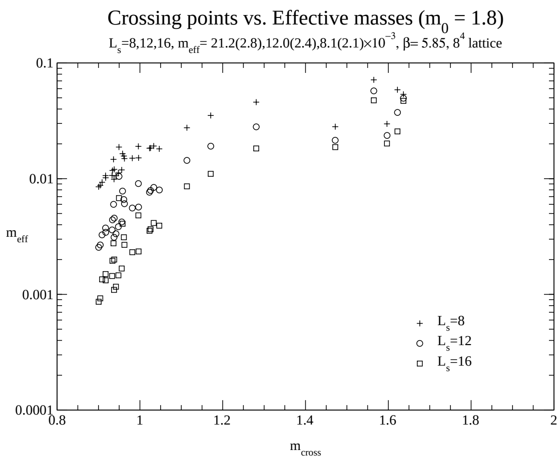

Figure 1. Effective quark mass induced by domain walls with domain wall height $m_0 = 1.8$ for the Monte Carlo configurations at $\beta = 5.85, 8^4$ lattice. L_s is the number of lattice sites in the fifth direction.

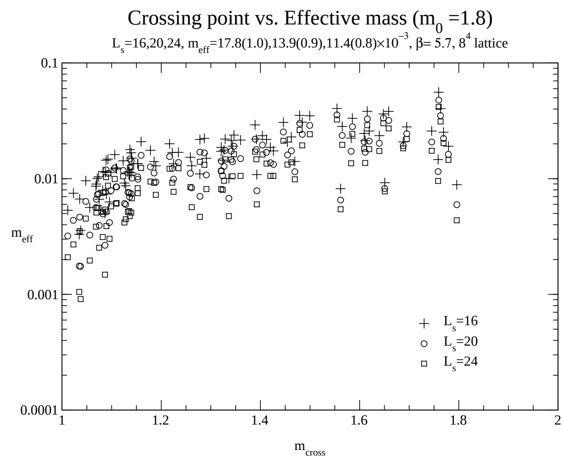

Figure 2. Effective quark mass induced by domain walls with domain wall height $m_0 = 1.8$ for the Monte Carlo configurations at $\beta = 5.7, 8^4$ lattice. L_s is the number of lattice sites in the fifth direction.

methods [7–9]. Here we report a more systematic study of the effective quark masses to understand the effects of different lattice sizes, the coupling constant β, and the form of gluon actions.

To study the effect of the gauge coupling, we have generated 100 $SU(3)$ lattice gauge configurations on a lattice size 8^4, 50 each at $\beta = 5.85$ and 5.7. We measure the spectral flow of the hermitian Wilson-Dirac operator to calculate the topological index of the gauge configurations, and calculate the eigenvalues of $H_{\rm DWF}(m_0 = 1.8)$ corresponding to the nontrivial topology of the gauge configurations. For larger lattice spacing, quantum fluctuations are stronger, and some of these fluctuations can be misidentified as small size instantons. It turns out that they can induce strong couplings between the left-and right-handed chiral modes and are detrimental to the existence of the low-energy effective theory. Indeed, for the same L_s, the effective masses are much larger at the smaller β's than those at $\beta = 6.0$. For example, with $L_s = 16$, $m_{\rm eff}$ is 0.00080(13) at $\beta = 6.0$, 0.008(2) at $\beta = 5.85$, and 0.018(1) at $\beta = 5.7$.

Moreover, for the same lattice size, physical volume is larger at smaller β, and hence can house more fermionic zero modes. As shown in Fig. 1 for $\beta = 5.85$ and Fig. 2 for $\beta = 5.70$, the total number of fermionic zero modes in the 50 configurations is now 32 and 96, respectively.

For $\beta = 5.7$, there are several crossings very close to $m_0 = 1.8$. This is in contrast to what has been observed at $\beta = 6.0$, where the crossings occur mostly around $m = 1.0$ [5,10]. The equal spacing (in a log plot) between $m_{\rm eff}$ at different L_s is a clear signal for the exponential decay. However, there is a significant variation in the rate among all the crossings, as evident in the figures. (Note: The recent works of the CP-PACS [7] and RBC [8] collaborations show the signal for varying rate of exponential decay and/or nonvanishing effective mass in the $L_s \to \infty$ limit. This behavior is only seen at a much larger L_s than those studied here. However, it is interesting to note that averaging eigenvalues with varying exponential rate can easily reproduce the large L_s behavior of the effective mass observed in the aforementioned reference.)

To see the volume dependence at a fixed m_0 and β, we also measure the effective mass on a set of 50 configurations on an $8^3 \times 16$ lattice at $\beta =$ 5.85. The total number of fermionic zero modes is now 64, doubling that on the 8^4 lattice. The

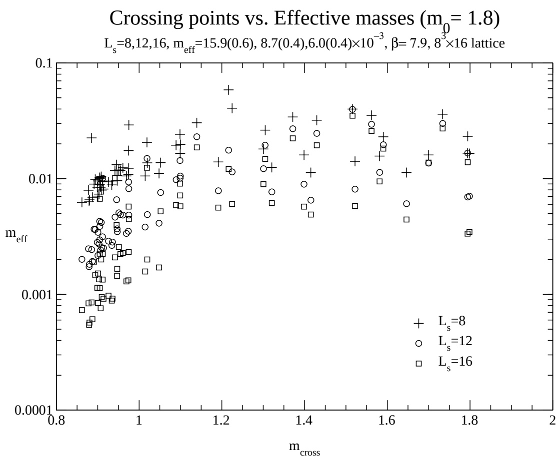

Figure 3. Effective quark mass induced by domain walls with domain wall height $m_0 = 1.8$ for the 50 Monte Carlo configurations generated by the 1-loop, tadpole improved gauge action [11] on a $\beta = 7.9, 8^3 \times 16$ lattice. L_s is the number of lattice sites in the fifth direction.

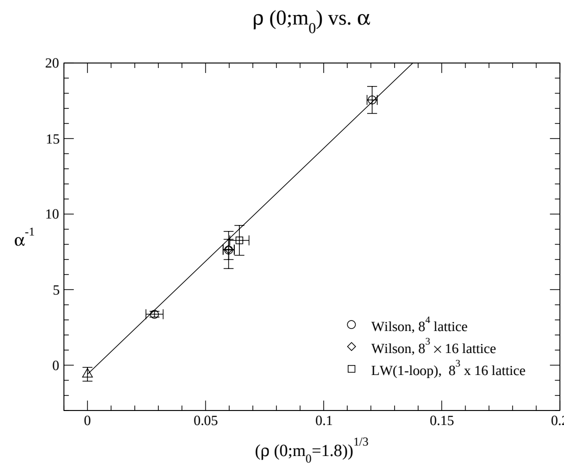

Figure 4. Average coefficient of the exponential decreases as a function of $\rho(0; m_0)$. The extrapolation of the fit to the continuum limit for the Wilson gauge backgrounds is shown on the left. $\rho(0; m_0)$ for the Wilson and improved gauge action are from Ref. [10] for $8^3 \times 16$ lattices.

average effective mass turns out to be essentially the same as that on the smaller lattice.

We also measure the effective mass on a set of 200 configurations generated from the 1-loop Lüscher-Weisz gauge action [11] with the tadpole improvement. (Crossings from only 50 lattices are shown in Fig. 3.) Similar studies using various improved gauge actions, including an RG improved action [12] are reported in [7,13]. The gauge coupling ($\beta = 7.9$) corresponds to a spacing of ~ 0.16 fm, similar to $\beta = 5.7$ of the Wilson action. The spectral flow and the domain-wall eigenvalues are studied with the same Wilson-Dirac operator. As shown in Fig. 3, the number of instantons as well as the distribution of the crossings differs significantly from the Wilson action at similar lattice spacing. Because of the decrease of small-scale quantum fluctuations, the probability of the crossing at $m_0 > 1.2$ is heavily suppressed, and the density of small eigenvalues of $H_{\rm W}(m_0)$ is also much smaller ($\sim 2.7 \times 10^{-4}$, compared to $\sim 1.8 \times 10^{-3}$ for $\beta = 5.7$ Wilson action). Therefore, the effective mass decreases faster as a function of L_s. The average effective mass at $L_s = 16$ is $\sim 6 \times 10^{-3}$, compared to 18×10^{-3} from the Wilson action at $\beta = 5.7$.

This point is also reflected in the dependence of the exponential decay rate on the density of the zero eigenvalues of $H_{\rm W}(m_0)$ ($\rho(0; m_0)$). We note that since the gauge fields are replicated along the fifth dimensional slices, the relevant length scale in the fifth dimension is the inverse of the rate of exponential decay (α) and by simple engineering dimensions is given qualitatively by the density of zero eigenvalues of $\rho(0; m_0)^{1/3}$. In Fig. 4, we have plotted $1/\alpha$ as a function of $\rho(0; m_0)$. The rate α for each coupling is calculated by fitting $m_{\rm eff}(L_s) = m_0 \exp[-\alpha L_s]$ at different L_s. The statistical errors are estimated by doing correlated fits to single-eliminated jackknife blocks. The inverse decay rates from all the configurations studied show an approximate linear scaling as $\rho(0; m_0)^{1/3}$. This suggests that the density of small eigenvalues is indeed the dominating factor for the exponential decay rate of domain-wall fermions.

Effective masses thus obtained for the different gauge coupling and L_s are plotted in Fig. 5.

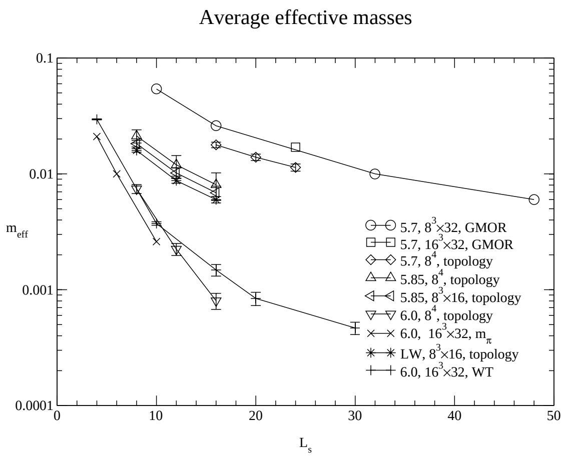

Figure 5. Average effective masses from various observables as a function of L_s. Effective masses from $\beta = 5.7$, GMOR relation are from Ref. [14]. Data for $\beta = 6.0$, m_π and the axial Ward–Takahashi(WT) identity are from Ref. [9] and [7], respectively. LW denotes the 1-loop, tadpole improved gauge action from Ref. [11].

The data from Ref. [9,14] are also included for comparison. For a given gauge coupling, different volumes and methods of measurements have little effect on the size of the effective mass as well as the rate of exponential decay. However, the change of gauge action affects the effective mass significantly. It is quite clear from the figure 5 that for a practical simulation of the domain-wall fermion, one either chooses a large β with the conventional Wilson action or an improved action keeping lattice spacing large.

To summarize, we have studied the residual chiral symmetry breaking present in domain-wall fermion by measuring the eigenvalues of the hermitian domain-wall Dirac operator corresponding to the topology of the lattice gauge configurations. Individual eigenvalues for the topological zero modes show clear exponential behavior in L_s. We regard these eigenvalues as the induced mass for the surface chiral modes at finite L_s separation.

For L_s and β, we see little variation of $m_{\rm eff}$ as a function of the volume. This is in some sense expected because the coupling of the chiral modes between the opposite walls has little to do with the size of the four-dimensional slice. On the other hand, a strong dependence on β is observed. In particular, the effective mass is much larger at $\beta = 5.85$ or 5.7 than that at $\beta = 6.0$. For the improved gauge action, the spurious fluctuations are reduced significantly and the L_s needed to obtain a good chiral symmetry is reduced. Since the additional computation needed for the improved action is negligible, using improved gluon actions may enable us to simulate domain-wall fermions with larger lattice spacing.

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